

QUIVER HALL-LITTLEWOOD FUNCTIONS AND KOSTKA-SHOJI POLYNOMIALS

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ABSTRACT. We define a family of Hall-Littlewood functions for any quiver. These form a basis for a tensor power of symmetric functions over a field with several parameters, one for each arrow in the quiver. For the single loop quiver, our functions are the usual (plethystically modified) Hall-Littlewood functions. For a cyclic quiver with more than one vertex, they are modified versions of multisymmetric functions defined by Shoji. The general quiver Hall-Littlewood functions are defined via creation operators and also admit a geometric interpretation in terms of equivariant Euler characteristics on Lusztig's iterated convolution diagrams. The latter construction is inspired by recent work of Finkelberg and Ionov in the context of cyclic quivers.

We conjecture that the quiver Hall-Littlewood functions are Schur-positive for arbitrary quivers. In the context of cyclic quivers we propose an explicit multitableaux formula for the multiparameter Kostka-Shoji polynomials of Finkelberg and Ionov.

1. INTRODUCTION

The Kostka-Foulkes polynomials $K_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ have numerous manifestations in combinatorics, representation theory of the symmetric and general linear groups, and geometry. For example, being special Kazhdan-Lusztig polynomials for the affine symmetric group, they describe intersection cohomology of Schubert varieties in the affine flag ind-variety. They also occur as graded multiplicities of modules for the general linear group obtained by twisting functions on the nullcone by a line bundle [B]. The positivity is exhibited combinatorially by the theorem of Lascoux and Schützenberger, who give the Kostka-Foulkes polynomial as the sum over semistandard Young tableaux of shape λ and weight μ , with grading given by the charge function [LS].

The Kostka-Shoji polynomials $K_{\lambda^\pm, \mu^\bullet}^\pm(q)$ of [Sh3] (see also [Sh1]) are a generalization of the Kostka polynomials indexed by a pair of r -multipartitions (r -tuples of partitions) where r is an arbitrary positive integer. From the point of view of [Sh1], these are associated with the complex reflection group $G(r, 1, n)$. For $r = 1$, this is nothing but S_n , and for $r = 2$ it is the Weyl group of type C_n ; accordingly, the Kostka-Shoji functions for $r = 2$ are related to Green functions for the group $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ (or $\mathrm{SO}_{2n+1}(\mathbb{F}_q)$).

Achar and Henderson [AH] realized the Kostka-Shoji polynomials for $r = 2$ as the Poincaré polynomials of intersection cohomology stalks of GL_n -orbit closures in the enhanced nilpotent cone $\mathbb{C}^n \times \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{gl}_n$ is the nilpotent cone of GL_n . By Finkelberg, Ginzburg, and Travkin [FGT], a similar realization holds for the $GL_n(\mathbb{C}[[z]])$ -orbit closures in the mirabolic affine Grassmannian of GL_n .

Another interpretation of the Kostka-Shoji polynomials was proposed by Finkelberg and Ionov [FI]. They define multivariable Kostka-Shoji polynomials as graded multiplicities of irreducible GL_n^r -modules (indexed by a multipartition λ^\bullet) in the derived global sections of an equivariant line bundle $\mathcal{O}(\mu^\bullet)$ over the total space of a vector bundle over Fl_n^r , which is an analogue of the cotangent bundle. For μ^\bullet regular, it is proved in [FI] that these polynomials are positive, that is, $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_1, \dots, q_r) \in \mathbb{Z}_{\geq 0}[q_1, \dots, q_r]$. They conjecture the positivity for all $\lambda^\bullet, \mu^\bullet$. Moreover, they conjecture that $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q, \dots, q)$ is equal to the Kostka-Shoji polynomial $K_{\lambda^\bullet, \mu^\bullet}^-(q)$. The latter was known to be true for $r = 1$ [Mac] and $r = 2$ [Sh2], and very recently the equality for all r was established by Shoji [Sh3, Theorem 7.3].

Fundamental in the approach of [FI] is the connection to Lusztig's iterated convolution diagram [L] for the cyclic quiver with r vertices; the convolution diagram is the GL_n^r -equivariant vector bundle over Fl_n^r mentioned above. Inspired by this connection, we define quiver Kostka-Shoji polynomials $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}^{Q, \hat{Q}}(q_{Q_1})$ for an arbitrary quiver $Q = (Q_0, Q_1)$ and an arbitrary acyclic subquiver \hat{Q} . These polynomials are indexed by a pair of Q_0 -tuples of partitions and they depend on parameters $q_{Q_1} = \{q_a \mid a \in Q_1\}$ indexed by directed edges of the quiver. The definition of the quiver Kostka-Shoji polynomials is a natural generalization of the construction from [FI] from a cyclic quiver to a general one. By construction, the quiver Kostka-Shoji polynomials are manifestly integral, i.e., they belong to $\mathbb{Z}[q_{Q_1}]$. We conjecture that they in fact belong to $\mathbb{Z}_{\geq 0}[q_{Q_1}]$ for any quiver Q and acyclic subquiver \hat{Q} .

We also introduce quiver Hall-Littlewood symmetric functions $H_{\mu^\bullet}^{Q, \hat{Q}}$, which form a basis for the Q_0 -fold tensor power of symmetric functions Λ , and we prove that the quiver Kostka-Shoji polynomials are the expansion coefficients of H_{μ^\bullet} into the tensor Schur basis. We define the H_{μ^\bullet} using quiver generalizations of the Garsia-Jing creation operators [J] [G] for (modified) Hall-Littlewood functions. In the case of the cyclic quiver with r vertices, our H_{μ^\bullet} are the images of Shoji's symmetric functions $Q_{\mu^\bullet}^+$ [Sh3] under a plethystic automorphism (in our construction, the sign \pm is replaced by the quiver orientation). These in turn specialize to the modified Hall-Littlewood functions H_μ [G] and Hall-Littlewood functions Q_μ [Mac], respectively, when $r = 1$. We relate our H_{μ^\bullet} to finite-variable Laurent polynomials including the Hall-Littlewood P -polynomials [Mac] as a special case. For cyclic quivers our Laurent polynomials are similar but not identical to Shoji's R -polynomials [Sh3] (see Remark 14).

We prove a Morris-type recurrence relation for the quiver Kostka-Shoji polynomials for general quivers. This provides a more efficient method of computation. In the case of the cyclic quiver with any number of vertices, we give an explicit multitableaux conjecture for the Kostka-Shoji polynomials (Conjecture 28). It involves a subtle multitableau invariant that we call the cascading catabolism type. We have an algorithm which conjecturally gives a sign-reversing involution which would prove the combinatorial formula by cancelling from the Morris recurrence, but we have had difficulties controlling the cascading catabolism type when the multitableaux are altered by the putative involution. In special cases (such as when all partitions have at most two parts) we can prove that the involution is well-defined. Previously known special cases of cyclic quiver Kostka-Shoji polynomials all take the form of being equal to ordinary Kostka polynomials; for example, for $r = 2$, parameters equal, with one partition of μ^\bullet empty [ShL, Theorem 3.12].

In a separate work we shall investigate the parabolic case [OS]. For the single loop quiver the parabolic quiver Kostka-Shoji polynomials are the generalized Kostka polynomials [SW] [SZ1].

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2. QUIVER HALL-LITTLEWOOD SERIES $\chi_{\mu}^{Q, \hat{Q}}$ AND KOSTANT PARTITIONS

2.1. Combinatorial data. Let $Q = (Q_0, Q_1)$ be a quiver (directed graph) with vertex set Q_0 and arrow set Q_1 . We also fix the data of an acyclic (no directed cycles) subquiver \hat{Q} with the same vertex set: $\hat{Q}_0 = Q_0$. For simplicity of exposition it is assumed that there is at most one directed edge from any vertex to any other in Q .¹ For $a \in Q_1$ we write $h(a), t(a) \in Q_0$ for its head and tail respectively. For any Q , we denote by Q' the opposite quiver obtained by reversing all edges.

2.2. A vector bundle $Z^{Q, \hat{Q}}$ on multiflags. The following is inspired by [FI] and is an instance of Lusztig's iterated convolution diagram [L, §1.5].

Let V_n be the standard GL_n -module, $V = \bigoplus_{i \in Q_0} V_n^{(i)}$ a Q_0 -graded vector space with dimension n at each vertex, $\text{Fl}_n^{Q_0} = \prod_{i \in Q_0} \text{Fl}(V_n^{(i)})$ the Q_0 -fold product of varieties of complete flags in V_n , and $E_n = \prod_{a \in Q_1} \text{Hom}_{\mathbb{C}}(V_n^{(t(a))}, V_n^{(h(a))})$ the space of Q -representations on V . Let $Z = Z^{Q, \hat{Q}, n} \subset \text{Fl}_n^{Q_0} \times E_n$ be the incidence variety whose points are given by the data $(F^{(i)} \mid i \in Q_0; \phi_a \in \text{Hom}(V_n^{(t(a))}, V_n^{(h(a))}) \mid a \in Q_1)$ such that

$$(1) \quad \phi_a(F_k^{(t(a))}) \subset \begin{cases} F_k^{(h(a))} & \text{if } a \in \hat{Q}_1 \\ F_{k-1}^{(h(a))} & \text{if } a \in Q_1 \setminus \hat{Q}_1 \end{cases} \quad \text{for all } 1 \leq k \leq n$$

where $F_k^{(i)}$ is the k -dimensional subspace of the flag $F^{(i)}$. The projection $Z \rightarrow \text{Fl}_n^{Q_0}$ gives Z the structure of a homogeneous $GL_n^{Q_0}$ -vector bundle. The other projection map $\text{Spr} : Z^{Q, \hat{Q}} \rightarrow E_n$ is a desingularization of its image [L]. If Q is a single loop then Spr is Springer's desingularization of the nullcone $\mathcal{N} \subset \mathfrak{gl}_n$.

2.3. Arrow scaling. The $|Q_1|$ -dimensional torus T^{Q_1} acts on Z such that the a -th copy of \mathbb{C}^* acts by scaling the map ϕ_a for $a \in Q_1$. We write $R(T^{Q_1}) = \mathbb{Z}[q_a^{\pm 1} \mid a \in Q_1]$, where the action of the a -th copy of \mathbb{C}^* has weight q_a^{-1} . The q_a are called arrow variables. We identify arrow variables for a quiver Q and its opposite Q' , so that for all $a = (i, j) \in Q_1$ we have $q_a = q_{a'}$ where $a' = (j, i) \in Q'_1$.

2.4. Z as a $GL_n^{Q_0}$ -homogeneous vector bundle. Since Z is a homogeneous $GL_n^{Q_0}$ -vector bundle on $\text{Fl}_n^{Q_0}$, it is uniquely determined by its fiber over the base-point. This fiber can be described explicitly as follows.

The natural $GL_n^{Q_0}$ -action on E_n is obtained by restriction of the adjoint action of $GL(V)$ on $\mathfrak{g} = \text{End}(V)$. Let E_1 be defined similarly to E_n except that at each $i \in Q_0$ the corresponding vector space is one-dimensional. The space E_1 may be identified with the subspace of $Q_0 \times Q_0$ matrices $M_{Q_0 \times Q_0}$ with basis given by

¹This is not a serious restriction since a Kostka-Shoji polynomial for a quiver with multiple edges is recovered by a substitution from the corresponding one in which the multiple edges are replaced by a single edge.

elementary matrices E_{ij} for $(i, j) \in Q_1$. Define \hat{E}_1 to be the analogous space for the acyclic subquiver \hat{Q} . Define the subspace $W = W^{Q, \hat{Q}, n} \subset E_n \cong \mathfrak{gl}(V_n) \otimes E_1$ by

$$W = (\mathfrak{n}_n \otimes E_1) \oplus (\mathfrak{h}_n \otimes \hat{E}_1)$$

where \mathfrak{n}_n is the nilradical of the standard upper triangular Borel subalgebra $\mathfrak{b}_n \subset \mathfrak{gl}(V_n)$, and $\mathfrak{h}_n \subset \mathfrak{b}_n$ is the Cartan subalgebra.

Under the identification of associative algebras

$$(2) \quad \mathfrak{g} \cong \mathfrak{gl}(V_n) \otimes M_{Q_0 \times Q_0},$$

W can be viewed as a subalgebra of a standard Borel subalgebra \mathfrak{b} of \mathfrak{g} . The subalgebra \mathfrak{b} is upper triangular with respect to the following ordered basis of $V_n \otimes \mathbb{C}^{Q_0}$. Since \hat{Q} is acyclic, there is a total order $<$ on Q_0 such that all arrows point from smaller to larger vertices. Fix such an order. Let $(\epsilon^{(i)} \mid i \in Q_0)$ be the ordered basis for \mathbb{C}^{Q_0} and (e_1, \dots, e_n) the ordered standard basis of V_n . The subalgebra W is strictly upper triangular with respect to the ordered basis $e_k \otimes \epsilon^{(i)}$ using the lex order $e_k \otimes \epsilon^{(i)} < e_\ell \otimes \epsilon^{(j)}$ if $k < \ell$ or if $k = \ell$ and $i < j$ in Q_0 . Then $W \subset \mathfrak{n}$ where \mathfrak{n} is the nilradical of \mathfrak{b} .

Let $R_+^{Q, \hat{Q}}$ be the set of positive roots of \mathfrak{g} whose root vectors form a basis of W . There are two kinds. Those in diagonal blocks have the form $e_k \otimes (\epsilon^{(i)} - \epsilon^{(j)})$ where $(i, j) \in \hat{Q}_1$ and $1 \leq k \leq n$ is arbitrary. Those in blocks above the diagonal have the form $e_k \otimes \epsilon^{(i)} - e_\ell \otimes \epsilon^{(j)}$ where $1 \leq k < \ell \leq n$ and $(i, j) \in Q_1$.

Let $\text{pr} : \mathbb{Z}^n \otimes \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}^{Q_0}$ be the projection defined by $\text{pr}(e_k \otimes \epsilon^{(i)}) = \epsilon^{(i)}$ for all $1 \leq k \leq n$ and $i \in Q_0$. For $\alpha \in R_+^{Q, \hat{Q}}$ we define

$$(3) \quad q_\alpha = q_a = q_{i,j}$$

where $\text{pr}(\alpha) = \epsilon^{(i)} - \epsilon^{(j)}$.

Example 1. Let Q be the quiver depicted below and \hat{Q} the acyclic subquiver given by removing the edge $(3, 0)$. For $n = 3$ the various spaces are depicted. We use the total order $0 < 1 < 2 < 3$ on Q_0 . $W^{Q, \hat{Q}}$ is depicted in two ways, first using the totally ordered basis as above, and then using the total order which groups the basis elements with fixed $i \in Q_0$ together.

$$\begin{array}{c} 0 \\ \swarrow \quad \searrow \\ 3 \quad \longleftarrow \quad 1 \\ \nwarrow \quad \searrow \\ 2 \end{array} \quad E_1 = \begin{bmatrix} & * & & \\ & & * & * \\ * & & & * \end{bmatrix} \quad \hat{E}_1 = \begin{bmatrix} * & & & \\ & * & * & \\ & & * & * \end{bmatrix}$$

$\begin{matrix} * & & & \\ & * & * & \\ & & * & \end{matrix}$	$\begin{matrix} * & & & \\ & * & * & \\ & & * & \\ * & & & \end{matrix}$	$\begin{matrix} * & & & \\ & * & * & \\ & & * & \\ * & & & \end{matrix}$	$\begin{matrix} & * & * & * \\ & & * & * \\ & & & * \end{matrix}$	$\begin{matrix} & & & \\ & & & \\ * & * & * & \\ & * & * & \\ & & * & \end{matrix}$	$\begin{matrix} & * & * & * \\ & & * & * \\ & & & * \end{matrix}$
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Let $\mathfrak{h}_{Q_0} \subset M_{Q_0 \times Q_0}$ be the Cartan subalgebra of diagonal matrices. Under (2), $\mathfrak{b}_n \otimes \mathfrak{h}_{Q_0}$ is identified with the Lie algebra of $B_n^{Q_0}$ in \mathfrak{g} ; the latter is embedded as upper triangular matrices in the diagonal blocks of the right-hand figure in the Example 1.

Lemma 2. $W^{Q, \hat{Q}}$ is stable under the adjoint action of $\mathfrak{b}_n \otimes \mathfrak{h}_{Q_0}$ on \mathfrak{g} .

Proof. The Lie bracket on \mathfrak{g} transported to $\mathfrak{gl}(V_n) \otimes M_{Q_0 \times Q_0}$ via (2) is given by

$$\begin{aligned} [x_1 \otimes y_1, x_2 \otimes y_2] &= x_1 x_2 \otimes y_1 y_2 - x_2 x_1 \otimes y_2 y_1 \\ &= [x_1, x_2] \otimes y_1 y_2 + x_2 x_1 \otimes [y_1, y_2]. \end{aligned}$$

The fact that \hat{Q} is a subquiver of Q then implies that $W^{Q, \hat{Q}}$ is stable under $\mathfrak{b}_n \otimes \mathfrak{h}_{Q_0}$. \square

By Lemma 2 we may define a $GL_n^{Q_0}$ -homogeneous vector bundle on $\text{Fl}_n^{Q_0}$:

$$\mathcal{W}^{Q, \hat{Q}} = GL_n^{Q_0} \times^{B_n^{Q_0}} W^{Q, \hat{Q}}.$$

The fiber of $Z^{Q, \hat{Q}}$ over the basepoint, as a $B_n^{Q_0}$ -module, is $W^{Q', \hat{Q}'}$ for the *opposite* quiver Q' and opposite acyclic subquiver \hat{Q}' . Hence there is an isomorphism $\mathcal{W}^{Q', \hat{Q}'} \rightarrow Z^{Q, \hat{Q}}$ given by $(g, v)B_n^{Q_0} \mapsto (gB_n^{Q_0}, gv)$ for $g \in GL_n^{Q_0}$ and $v \in W^{Q', \hat{Q}'}$.

2.5. Twisting \mathcal{W} by a line bundle. For μ^\bullet a Q_0 -tuple of integral GL_n -weights $\mu^{(i)} \in \mathbb{Z}^n$, let $\mathcal{L}_{\mu^\bullet}$ be the $GL_n^{Q_0}$ -equivariant line bundle on $\text{Fl}_n^{Q_0}$ of weight $w_0^\bullet(\mu^\bullet)$ over the basepoint. Here w_0^\bullet is the long element in the Weyl group of $GL_n^{Q_0}$, which is the product of symmetric groups $S_n^{Q_0}$. Let $p : \mathcal{W} \rightarrow \text{Fl}_n^{Q_0}$ be the bundle map and $D_{w_0^\bullet}$ the K -theoretic Demazure operator. Let $\mathcal{G} = GL_n^{Q_0} \times T^{Q_1}$ and $\mathcal{T} = T_n^{Q_0} \times T^{Q_1}$, where $T_n \subset GL_n$ is the maximal torus of diagonal matrices. Let $x^{\mu^\bullet} \in R(T_n^{Q_0}) = \mathbb{Z}[(x_k^{(i)})^{\pm 1} \mid i \in Q_0, 1 \leq k \leq n]$ be defined by

$$x^{\mu^\bullet} = \prod_{\substack{i \in Q_0 \\ 1 \leq k \leq n}} (x_k^{(i)})^{\mu_k^{(i)}}.$$

We define the *quiver Hall-Littlewood series* $\chi_{\mu^\bullet}^{Q, \hat{Q}}$ to be the \mathcal{G} -equivariant Euler characteristic of the global sections functor applied to $p^*(\mathcal{L}_{\mu^\bullet})$:

$$\begin{aligned}
\chi_{\mu^\bullet}^{Q, \hat{Q}} &= \text{ch}_{\mathcal{G}} \sum_{i \geq 0} (-1)^i H^i(\mathcal{W}^{Q', \hat{Q}'}, p^* \mathcal{L}_{\mu^\bullet}) \\
&= \text{ch}_{\mathcal{G}} \sum_{i \geq 0} (-1)^i H^i(\text{Fl}_n^{Q_0}, \mathcal{L}_{\mu^\bullet} \otimes \text{Sym}((\mathcal{W}^{Q', \hat{Q}'})^\vee)) \\
&= D_{w_0^\bullet} x^{\mu^\bullet} \prod_{\alpha \in R_+^{Q, \hat{Q}}} (1 - \text{ch}_{\mathcal{T}}(\alpha))^{-1} \\
&= D_{w_0^\bullet} x^{\mu^\bullet} B(x; q_{Q_1}) \quad \text{where} \\
B(x; q_{Q_1}) &= \prod_{\substack{(i,j) \in \hat{Q}_1 \\ 1 \leq k \leq \ell \leq n}} \frac{1}{1 - q_{i,j} x_k^{(i)} / x_\ell^{(j)}} \prod_{\substack{(i,j) \in Q_1 \setminus \hat{Q}_1 \\ 1 \leq k < \ell \leq n}} \frac{1}{1 - q_{i,j} x_k^{(i)} / x_\ell^{(j)}}.
\end{aligned}$$

This is understood as an element of $R(T_n^{Q_0})[[q_{Q_1}]]$, i.e., as the character of a T^{Q_1} -graded locally finite $GL_n^{Q_0}$ -module. Note that while we applied the Euler characteristic on $\mathcal{W}^{Q', \hat{Q}'}$ we end up computing with the positive roots $R_+^{Q, \hat{Q}}$ for Q .

2.6. Quiver Kostka-Shoji polynomials. For $\lambda^\bullet \in X_+^{Q_0}$ a Q_0 -tuple of dominant integral GL_n -weights, we define the *quiver Kostka-Shoji polynomials* $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) \in \mathbb{Z}[q_{Q_1}]$ by the expansion

$$(4) \quad \chi_{\mu^\bullet}^{Q, \hat{Q}} = \sum_{\lambda^\bullet \in X_+^{Q_0}} \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) s_{\lambda^\bullet}(x)$$

of the quiver Hall-Littlewood series into products of Schur polynomials $s_{\lambda^\bullet}(x) = \prod_{i \in Q_0} s_{\lambda^{(i)}}(x_1^{(i)}, \dots, x_n^{(i)})$. The fact that $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) \in \mathbb{Z}[q_{Q_1}]$ is justified by the Kostant partition formula (5) below.

Let $\Delta^\bullet(x) = \prod_{i \in Q_0} \Delta^{(i)}(x)$ where $\Delta^{(i)}(x) = \prod_{1 \leq k < \ell \leq n} (x_k^{(i)} - x_\ell^{(i)})$ is the Vandermonde in the i -th set of x variables. Let $J = \sum_{w \in S_n} (-1)^w w$ be the antisymmetrizer over the symmetric group and J^\bullet the antisymmetrizer for $S_n^{Q_0}$. Let $\rho = (n-1, n-2, \dots, 1, 0) \in \mathbb{Z}^n$ and $\rho^\bullet \in (\mathbb{Z}^n)^{Q_0}$ the Q_0 -tuple such that $\rho^{(i)} = \rho$ for all $i \in Q_0$. For one set of variables (z_1, \dots, z_n) we have $D_{w_0}(f) = J(z^\rho)^{-1} J(z^\rho f)$.

We use the notation $[f]g$ for the coefficient of f in g . We have

$$\begin{aligned}
\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) &= [s_{\lambda^\bullet}(x)] D_{w_0^\bullet} x^{\mu^\bullet} B(x; q_{Q_1}) \\
&= [s_{\lambda^\bullet}(x)] \Delta^\bullet(x)^{-1} J^\bullet(x^{\mu^\bullet + \rho^\bullet} B(x; q_{Q_1})) \\
&= [J^\bullet(x^{\lambda^\bullet + \rho^\bullet})] J^\bullet(x^{\mu^\bullet + \rho^\bullet} B(x; q_{Q_1})) \\
&= [x^{\lambda^\bullet + \rho^\bullet}] J^\bullet(x^{\mu^\bullet + \rho^\bullet} B(x; q_{Q_1})) \\
&= \sum_{w^\bullet \in S_n^{Q_0}} (-1)^{w^\bullet} [x^{\lambda^\bullet + \rho^\bullet - w^\bullet(\mu^\bullet + \rho^\bullet)}] B(w^\bullet x; q_{Q_1}) \\
&= \sum_{w^\bullet \in S_n^{Q_0}} (-1)^{w^\bullet} [x^{w^{\bullet-1}(\lambda^\bullet + \rho^\bullet) - (\mu^\bullet + \rho^\bullet)}] B(x; q_{Q_1}).
\end{aligned}$$

Therefore

$$(5) \quad \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) = \sum_{w^\bullet \in S_n^{Q_0}} (-1)^{w^\bullet} \sum_{m: R_+^{Q, \hat{Q}} \rightarrow \mathbb{Z}_{\geq 0}} \prod_{\alpha \in R_+^{Q, \hat{Q}}} q_\alpha^{m(\alpha)}$$

where the Kostant partition m satisfies

$$(6) \quad \sum_{\alpha \in R_+^{Q, \hat{Q}}} m(\alpha) \alpha = (w^\bullet)^{-1}(\lambda^\bullet + \rho^\bullet) - (\mu^\bullet + \rho^\bullet).$$

Conjecture 3. *For any quiver Q and acyclic subquiver \hat{Q} , the quiver Kostka-Shoji polynomial $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ has nonnegative integer coefficients.*

Example 4. Consider the equioriented cyclic quiver with r vertices, that is, $Q_0 = \mathbb{Z}/r\mathbb{Z}$ and arrows $(i, i+1)$ for $i \in Q_0$ and indices taken mod r . Let \hat{Q} be obtained by removing the arrow $(r-1, 0)$. Then $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ is the Finkelberg-Ionov polynomial.² In [FI] positivity is proved for μ^\bullet regular.

When $r = 2$ and every arrow parameter is set to a single parameter q , the quiver Kostka-Shoji polynomial is the Shoji polynomial $K_{\lambda^\bullet, \mu^\bullet}^-(q)$. When $r = 1$ (single loop quiver with \hat{Q} empty) the quiver Kostka-Shoji polynomial is the Kostka-Foulkes polynomial [Mac].

2.7. Dominance. Choose a total order $<$ on Q_0 which refines the partial order defined by the acyclic subquiver \hat{Q} : $i < j$ for all $(i, j) \in \hat{Q}_1$. Let X^{Q_0} be the set of Q_0 -tuples $\mu^\bullet = (\mu^{(i)} \mid i \in Q_0)$ of elements $\mu^{(i)}$ of the weight lattice $X \cong \mathbb{Z}^n$ of \mathfrak{gl}_n .

For clarity of notation, in the following definition we will assume that Q_0 has vertices $0 < 1 < \dots < r-1$. Let $j : X^{Q_0} \rightarrow \mathbb{Z}^{n|Q_0|}$ be the linear map defined by

$$j(\mu^\bullet) = (\mu_1^{(0)}, \mu_1^{(1)}, \dots, \mu_1^{(r-1)}, \mu_2^{(0)}, \mu_2^{(1)}, \dots, \mu_2^{(r-1)}, \mu_3^{(0)}, \dots, \mu_n^{(r-1)}).$$

We regard $\mathbb{Z}^{n|Q_0|}$ as the weight lattice $X(\mathfrak{g})$ of \mathfrak{g} . With the above choices the roots $R_+^{Q, \hat{Q}}$ are positive roots for the standard Borel in \mathfrak{g} , as in the first of the two pictures of $R_+^{Q, \hat{Q}}$ in Example 1.

The dominance order $\lambda^\bullet \succeq \mu^\bullet$ on X^{Q_0} is defined by

$$j(\lambda^\bullet) - j(\mu^\bullet) \in Q^+(\mathfrak{g})$$

where $Q^+(\mathfrak{g})$ is the positive cone of roots in \mathfrak{g} . The map j will often be suppressed in the notation.

Lemma 5. $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ is zero unless $\lambda^\bullet \succeq \mu^\bullet$.

Proof. Using (5) we see that $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ is zero unless there is a $w^\bullet \in S_n^{Q_0}$ such that $w^{\bullet-1}(\lambda^\bullet + \rho^\bullet) - (\mu^\bullet + \rho^\bullet) \in Q^+(\mathfrak{g})$. Since $\lambda^{(i)}$ is dominant for each $i \in Q_0$, it follows that $(\lambda^{(i)} + \rho^{(i)}) - (w^{(i)})^{-1}(\lambda^{(i)} + \rho^{(i)}) \in Q^+(\mathfrak{gl}_n)$ for all $i \in Q_0$. This implies that $(\lambda^\bullet + \rho^\bullet) - w^{\bullet-1}(\lambda^\bullet + \rho^\bullet) \in Q^+(\mathfrak{g})$. We obtain $\lambda^\bullet - \mu^\bullet \in Q^+(\mathfrak{g})$. \square

²Their parameters are labeled by the tail of the arrow of our parameters.

3. QUIVER HALL-LITTLEWOOD SYMMETRIC FUNCTIONS VIA CREATION OPERATORS

We now develop a quiver analogue of Garsia-Jing creation operators [J] [G] for the Hall-Littlewood H -functions (also known as the Q' -functions), which are the plethystic cousins of the Hall-Littlewood Q -functions [Mac]. We will make extensive use of plethystic operations on symmetric functions. Our conventions follow those of [LR, SZ1, SZ2]. We refer the reader unfamiliar with plethysm in the context of symmetric functions to [LR] for a thorough treatment. The main definition is that Ω is the generating function of single row Schur functions:

$$\Omega[X] = \sum_{r \geq 0} h_r[X], \quad h_r = s_{(r)}.$$

Many of our formulas can be understood in concrete terms by using the following properties of Ω :

$$\begin{aligned} \Omega[X + Y] &= \Omega[X]\Omega[Y], & \Omega[-X] &= 1/\Omega[X] \\ \Omega[z] &= 1/(1 - z), & \Omega[-z] &= (1 - z) \end{aligned}$$

where X, Y are any plethystic alphabets and z is a single variable or monomial.

3.1. Quiver symmetric function space with automorphism. For any quiver $Q = (Q_0, Q_1)$, define the algebra of *quiver symmetric functions* as

$$\Lambda_Q = \bigotimes_{i \in Q_0} \Lambda^{(i)}$$

where $\Lambda = \Lambda_F$ is the algebra of symmetric functions [Mac] over $F = \text{Frac}(R(T^{Q_1})) = \mathbb{Q}(q_a \mid a \in Q_1)$ and each $\Lambda^{(i)}$ is a copy of Λ with its own alphabet $X^{(i)} = x_1^{(i)} + x_2^{(i)} + \dots$. The algebra Λ_Q is graded by the set $\mathbb{Z}_{\geq 0}^{Q_0}$ of effective dimension vectors. Let Ψ be the automorphism of Λ_Q given by the plethystic substitutions

$$(7) \quad \Psi : X^{(j)} \mapsto X^{(j)} - \sum_{\substack{a \in Q_1 \\ h(a)=j}} q_a X^{(t(a))} \quad \text{for } j \in Q_0.$$

Explicitly, if $p_k^{(j)} = 1 \otimes \dots \otimes 1 \otimes p_k \otimes 1 \otimes \dots \otimes 1$ is the power sum embedded at the j -th tensor position for $j \in Q_0$, we have

$$\Psi(p_k^{(j)}) = p_k^{(j)} - \sum_{\substack{a \in Q_1 \\ h(a)=j}} q_a^k p_k^{(t(a))} \quad \text{for } j \in Q_0.$$

Let $A = A^Q$ be the arrow matrix of Q , the $Q_0 \times Q_0$ matrix

$$(8) \quad A_{ij} = \sum_{\substack{a \in Q_1 \\ (i,j)=(t(a),h(a))}} q_a.$$

Then $I - A$ is the matrix defining the substitutions (7).

The matrix $I - A$ is invertible over F as its determinant is a polynomial with constant term 1. The inverse matrix $B = (I - A)^{-1}$ defines the inverse of Ψ :

$$\Psi^{-1} : X^{(j)} \mapsto \sum_{i \in Q_0} X^{(i)} B_{ij}.$$

The coefficient B_{ij} is the generating function of directed paths from i to j , where the weight of a path is the product of q_a as a runs over the arrows in the path.

Example 6. For the Jordan quiver, the algebra of quiver symmetric functions is just Λ_F , and we have a single alphabet X . Then Ψ is given by the familiar plethystic substitution $X \mapsto (1-q)X$, where q is the parameter for the single loop edge. Thus $A = 1 - q$ and $B = (1 - q)^{-1}$.

3.2. Quiver Cauchy kernel. The algebra Λ_Q affords a Hall inner product $\langle \cdot, \cdot \rangle$, induced by the Hall inner products on the tensor factors. Bases of Λ_Q are indexed by the set \mathbb{Y}^{Q_0} of Q_0 -tuples of partitions. An example is the tensor Schur basis $s_{\lambda^\bullet}[X^\bullet] = \otimes_{i \in Q_0} s_{\lambda^{(i)}}[X^{(i)}]$ for $\lambda^\bullet \in \mathbb{Y}^{Q_0}$. The bases $\{a_{\mu^\bullet} \mid \mu^\bullet \in \mathbb{Y}^{Q_0}\}$ and $\{b_{\mu^\bullet}\}$ of Λ_Q are dual bases with respect to $\langle \cdot, \cdot \rangle$ if and only if

$$\sum_{\mu^\bullet \in \mathbb{Y}^{Q_0}} a_{\mu^\bullet}(X^\bullet) b_{\mu^\bullet}(Y^\bullet) = \Omega \left[\sum_{i \in Q_0} X^{(i)} Y^{(i)} \right]$$

The algebra Λ_Q has another scalar product $\langle \cdot, \cdot \rangle_Q$ defined by the *quiver Cauchy kernel* Ω_Q , which is the twist of the above Cauchy kernel by the automorphism Ψ acting on the X variables:

$$\begin{aligned} \Omega_Q &= \Omega \left[\sum_{i \in Q_0} X^{(i)} Y^{(i)} - \sum_{a \in Q_1} q_a X^{(t(a))} Y^{(h(a))} \right] \\ &= \Omega \left[\sum_{i \in Q_0} \Psi(X^{(i)}) Y^{(i)} \right]. \end{aligned}$$

Therefore the following are equivalent:

- (1) $\{a_{\mu^\bullet} \mid \mu^\bullet \in \mathbb{Y}^{Q_0}\}$ and $\{b_{\mu^\bullet}\}$ are dual bases with respect to $\langle \cdot, \cdot \rangle$.
- (2) $\{\Psi(a_{\mu^\bullet}) \mid \mu^\bullet \in \mathbb{Y}^{Q_0}\}$ and $\{b_{\mu^\bullet}\}$ are dual bases with respect to $\langle \cdot, \cdot \rangle_Q$.

3.3. The small quiver HL symmetric functions $H_{\mu^\bullet}^\varnothing[X^\bullet; q_{Q_1}]$. The following construction depends only on the quiver Q , not the choice of an acyclic subquiver \hat{Q} . Let u^\bullet be a Q_0 -tuple of auxiliary variables $u^{(i)}$ for $i \in I_0$. Write $u^{d^\bullet} = \prod_{i \in Q_0} (u^{(i)})^{d^{(i)}}$. For $d^\bullet \in \mathbb{Z}^{Q_0}$, the *small quiver Garsia-Jing creation operators* $H_{d^\bullet}^\varnothing \in \text{End}(\Lambda_Q)$ are the degree d^\bullet operators defined by the generating function

$$\begin{aligned} H^\varnothing(u^\bullet) &= \sum_{d^\bullet \in \mathbb{Z}^{Q_0}} u^{d^\bullet} H_{d^\bullet}^\varnothing \\ &= \Omega \left[\sum_{j \in Q_0} u^{(j)} X^{(j)} \right] \Omega \left[\sum_{(i,j) \in Q_0^2} (u^{(i)})^{-1} (A_{ij} - \delta_{ij}) X^{(j)} \right]^\perp \end{aligned}$$

where f^\perp denotes the adjoint to multiplication by $f \in \Lambda_Q$ with respect to the Hall scalar product $\langle \cdot, \cdot \rangle$.

For fixed $(i, j) \in Q_0^2$, the (i, j) -th skewing operator has the expansion

$$\Omega[(u^{(i)})^{-1} (A_{ij} - \delta_{ij}) X^{(j)}]^\perp = \sum_{m_{ij} \geq 0} (u^{(i)})^{-m_{ij}} h_{m_{ij}} [(A_{ij} - \delta_{ij}) X^{(j)}]^\perp$$

using a single index $m_{ij} \in \mathbb{Z}_{\geq 0}$. For a function $m : Q_0^2 \rightarrow \mathbb{Z}_{\geq 0}$ and $i \in Q_0$ let $|m(i, \bullet)| = \sum_{j \in Q_0} m(i, j)$. We obtain

$$H_{d^\bullet}^\varnothing = \sum_{m: Q_0^2 \rightarrow \mathbb{Z}_{\geq 0}} \prod_{i \in Q_0} h_{d^{(i)} + |m(i, \bullet)|} [X^{(i)}] \prod_{(i, j) \in Q_0^2} h_{m(i, j)} [(A_{ij} - \delta_{ij}) X^{(j)}]^\perp.$$

3.4. Quiver HL symmetric functions $H_{\mu^\bullet}[X^\bullet; q_{Q_1}]$. For this construction we require the additional data of an acyclic subquiver \hat{Q} of Q . Let $\hat{A} = A^{\hat{Q}}$ be the arrow matrix (8) of \hat{Q} . The quiver Garsia-Jing creation operator H_{d^\bullet} is defined by

$$H(u^\bullet) = \sum_{d^\bullet \in \mathbb{Z}^{Q_0}} u^{d^\bullet} H_{d^\bullet} = \Omega \left[\sum_{(i, j) \in Q_0^2} \frac{u^{(j)}}{u^{(i)}} \hat{A}_{ij} \right] H^\varnothing(u^\bullet).$$

By definition we have the infinite expansion

$$H_{d^\bullet} = \sum_{b: \hat{Q}_1 \rightarrow \mathbb{Z}_{\geq 0}} \prod_{a \in \hat{Q}_1} q_a^{b(a)} H_{d^\bullet + \text{wt}^{Q_0}(b)}^\varnothing \quad \text{for all } d^\bullet \in \mathbb{Z}^{Q_0}$$

where

$$\text{wt}^{Q_0}(b) = \sum_{a \in \hat{Q}_1} b(a) (\epsilon^{(t(a))} - \epsilon^{(h(a))}) \in \mathbb{Z}^{Q_0}.$$

The action of H_{d^\bullet} on Λ_Q is well-defined due to grading.

Remark 7. If the subquiver \hat{Q} has no edges then $H(u^\bullet) = H^\varnothing(u^\bullet)$.

Let $\mu^\bullet \in (\mathbb{Z}^n)^{Q_0}$ be a Q_0 -tuple of n -tuples of integers $\mu^{(i)} \in \mathbb{Z}^n$ for $i \in Q_0$. This can also be viewed as a sequence $\mu_1^\bullet, \mu_2^\bullet, \dots, \mu_n^\bullet$ of virtual dimension vectors $\mu_j^\bullet \in \mathbb{Z}^{Q_0}$ for $1 \leq j \leq n$. For any $\mu^\bullet \in (\mathbb{Z}^n)^{Q_0}$ the *quiver Hall-Littlewood symmetric function* $H_{\mu^\bullet}[X^\bullet; q_{Q_1}] = H_{\mu^\bullet}^{Q, \hat{Q}}[X^\bullet; q_{Q_1}] \in \Lambda_Q$ is defined by

$$H_{\mu^\bullet}[X^\bullet; q_{Q_1}] = H_{\mu_1^\bullet} H_{\mu_2^\bullet} \cdots H_{\mu_n^\bullet} \cdot 1.$$

Proposition 8. *The $H_{\mu^\bullet}[X^\bullet; q_{Q_1}]$ are a basis of Λ_Q for $\mu^\bullet \in \mathbb{Y}^{Q_0}$.*

Proof. By definition we see that $H_{d^\bullet}|_{q_a=0}$ is a Q_0 -tensor of Bernstein creation operators for Schur functions. Therefore $H_{\mu^\bullet}[X^\bullet; 0] = s_{\mu^\bullet}[X^\bullet]$, which is the tensor Schur basis of Λ_Q . It follows that the $H_{\mu^\bullet}[X^\bullet; q_{Q_1}]$ form a basis. \square

Next we consider the expansion of quiver Hall-Littlewood functions into the Schur basis. The transition coefficients are the quiver Kostka-Shoji polynomials from (4).

Theorem 9. *For every $\mu^\bullet \in \mathbb{Y}^{Q_0}$*

$$H_{\mu^\bullet}[X^\bullet; q_{Q_1}] = \sum_{\lambda^\bullet \in \mathbb{Y}^{Q_0}} \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) s_{\lambda^\bullet}[X^\bullet].$$

The proof of Theorem 9 will be given in Section 3.5.

Remark 10. (1) In general the series $\chi_{\mu^\bullet}^{Q, \hat{Q}}$ has infinitely many irreducible $GL_n^{Q_0}$ -characters $s_{\lambda^\bullet}(u)$ appearing with nonzero coefficients $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ with λ^\bullet having arbitrarily negative integer parts. The above theorem says that if one takes the polynomial part of the character $\chi_{\mu^\bullet}^{Q, \hat{Q}}$ (i.e., one retains only

terms for s_{λ^\bullet} where λ^\bullet is a sequence of partitions, that is, dominant integral weights with all parts nonnegative) and replaces the Schur polynomials $s_{\lambda^{(i)}}(x_1^{(i)}, \dots, x_n^{(i)})$ by symmetric functions $s_{\lambda^{(i)}}[X^{(i)}]$ then one obtains $H_{\mu^\bullet}[X^\bullet; q_{Q_1}]$.

- (2) Since $D_{w_0^\bullet}$ commutes with multiplication by the determinant character $\det^{(i)} = \prod_{k=1}^n x_k^{(i)}$ for all $i \in Q_0$, it follows that

$$\mathcal{K}_{\lambda^\bullet + r^\bullet, \mu^\bullet + r^\bullet}(q_{Q_1}) = \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$$

where by abuse of notation, $\lambda^\bullet + r^\bullet$ denotes the Q_0 -tuple of GL_n -weights whose i -th component is the result of adding $r^{(i)} \in \mathbb{Z}$ to each part of $\lambda^{(i)}$.

Therefore every coefficient $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ appearing in the series $\chi_{\mu^\bullet}^{Q, \hat{Q}}$ occurs as a coefficient in some symmetric function H_{ν^\bullet} .

- (3) If $n = 1$ then μ^\bullet is a single virtual dimension vector $d^\bullet \in \mathbb{Z}^{Q_0}$ and

$$(9) \quad H_{d^\bullet}[X^\bullet; q_{Q_1}] = \sum_{b: \hat{Q}_1 \rightarrow \mathbb{Z}_{\geq 0}} \prod_{a \in \hat{Q}_1} q_a^{b(a)} h_{d^\bullet + \text{wt } Q_0(b)}[X^\bullet].$$

The sum is finite since $h_r[X] = 0$ for $r < 0$ and \hat{Q} is acyclic.

Example 11. For the cyclic quiver and acyclic subquiver as in Example 4 with $r = 2$ and $\mu^\bullet = (\mu^{(0)}, \mu^{(1)}) = ((1, 1), (2))$, we have

$$\begin{aligned} H_{((1,1),(2))} &= s_{((1,1),(2))} + q_{10} s_{((1),(3))} + q_{01} s_{(2,1),(1)} + q_{01} q_{10} s_{((2),(2))} \\ &\quad + q_{01}^2 s_{((3,1), \emptyset)} + q_{01}^2 q_{10} s_{((3),(1))} + q_{01}^3 q_{10} s_{((4), \emptyset)}. \end{aligned}$$

3.5. Commutation of creation operators. To compute $H_{\mu^\bullet}[X^\bullet; q_{Q_1}]$ we use the commutation relations to relate the composition of creation operators with its “normal ordering” which has all multiplication operators on the left and all skewing operators on the right. Let $u_k^{(i)}$ be variables for $1 \leq k \leq n$ and $i \in Q_0$. The k -th Q_0 -tuple of variables is denoted u_k^\bullet .

Lemma 12.

$$H^\varnothing(u_1^\bullet) H^\varnothing(u_2^\bullet) \cdots H^\varnothing(u_n^\bullet) =$$

$$\Omega \left[\sum_{\substack{(i,j) \in Q_0^2 \\ 1 \leq k < \ell \leq n}} \frac{u_\ell^{(j)}}{u_k^{(i)}} (A_{ij} - \delta_{ij}) \right] \Omega \left[\sum_{\substack{p \in Q_0 \\ 1 \leq k \leq n}} u_k^{(p)} X^{(p)} \right] \Omega \left[\sum_{\substack{(i,j) \in Q_0^2 \\ 1 \leq k \leq n}} \frac{1}{u_k^{(i)}} (A_{ij} - \delta_{ij}) X^{(j)} \right]^\perp.$$

Proof. All skewing operators mutually commute and all multiplication operators mutually commute. For all $1 \leq k < \ell \leq n$ the k -th skewing operator must be commuted to the right past the ℓ -th multiplication operator. Such operators commute if they act on different sets of X -variables. Otherwise a constant factor is produced for each pair of out-of-order noncommuting operators according to the relation (see, e.g., [SZ2, (7)]):

$$\Omega[wX]^\perp \Omega[zX] = \Omega[zw] \Omega[zX] \Omega[wX]^\perp. \quad \square$$

Now we are ready to prove Theorem 9.

Proof of Theorem 9. Applying the above composite operator to 1 yields

$$H^\varnothing(u_1^\bullet)H^\varnothing(u_2^\bullet)\cdots H^\varnothing(u_n^\bullet)\cdot 1 = \Omega \left[\sum_{\substack{(i,j)\in Q_0^2 \\ 1\leq k<\ell\leq n}} \frac{u_\ell^{(j)}}{u_k^{(i)}} (A_{ij} - \delta_{ij}) \right] \Omega \left[\sum_{\substack{p\in Q_0 \\ 1\leq k\leq n}} u_k^{(p)} X^{(p)} \right].$$

Let \mathbb{Y}_n be the set of partitions with at most n parts. Using

$$\prod_{\substack{1\leq k<\ell\leq n \\ (i,j)\in Q_0^2}} \Omega \left[-\frac{u_\ell^{(j)}}{u_k^{(i)}} \delta_{ij} \right] = u^{-\rho^\bullet} \Delta^\bullet(u)$$

and the Cauchy formula we have

$$\begin{aligned} & u^{\rho^\bullet} H^\varnothing(u_1^\bullet)H^\varnothing(u_2^\bullet)\cdots H^\varnothing(u_n^\bullet)\cdot 1 \\ &= \Delta^\bullet \Omega \left[\sum_{\substack{(i,j)\in Q_0^2 \\ 1\leq k<\ell\leq n}} \frac{u_\ell^{(j)}}{u_k^{(i)}} A_{ij} \right] \sum_{\lambda^\bullet \in \mathbb{Y}_n^{Q_0}} s_{\lambda^\bullet}[u_1^\bullet, \dots, u_n^\bullet] s_{\lambda^\bullet}[X^\bullet]. \end{aligned}$$

Multiplying by $\Omega[\sum_{(i,j)\in Q_0^2} \sum_{k=1}^n (u_k^{(i)})^{-1} u_k^{(j)} \hat{A}_{ij}]$ we obtain

$$u^{\rho^\bullet} H(u_1^\bullet)H(u_2^\bullet)\cdots H(u_n^\bullet)\cdot 1 = B(u; q_{Q_1})^* \sum_{\lambda^\bullet \in \mathbb{Y}^{Q_0}} J^\bullet(u^{\lambda^\bullet + \rho^\bullet}) s_{\lambda^\bullet}[X^\bullet]$$

where $B(u; q_{Q_1})^*$ is $B(u; q_{Q_1})$ but with every $u_k^{(i)}$ replaced by its reciprocal. Taking the coefficient of $s_{\lambda^\bullet}[X^\bullet] u^{\mu^\bullet + \rho^\bullet}$ we have

$$\begin{aligned} [s_{\lambda^\bullet}[X^\bullet]] H_{\mu^\bullet}(X^\bullet; q_{Q_1}) &= [u^{\mu^\bullet + \rho^\bullet}] J^\bullet(u^{\lambda^\bullet + \rho^\bullet}) B(u; q_{Q_1})^* \\ &= \sum_{w^\bullet \in S_n^{Q_0}} (-1)^{w^\bullet} [u^{\mu^\bullet + \rho^\bullet - w^\bullet(\lambda^\bullet + \rho^\bullet)}] B(u; q_{Q_1})^* \\ &= \sum_{w^\bullet \in S_n^{Q_0}} (-1)^{w^\bullet} [u^{w^\bullet(\lambda^\bullet + \rho^\bullet) - (\mu^\bullet + \rho^\bullet)}] B(u; q_{Q_1}) \\ &= \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}). \end{aligned} \quad \square$$

3.6. Plethystic cousins. We define quiver Hall-Littlewood functions Q_{μ^\bullet} as the images of H_{μ^\bullet} under the automorphism Ψ of (7):

$$Q_{\mu^\bullet}[X^\bullet; q_{Q_1}] = \Psi(H_{\mu^\bullet}[X^\bullet; q_{Q_1}]).$$

Creation operators for Q_{μ^\bullet} are found by conjugating $H(u^\bullet)$ by Ψ . Explicitly, for $\mu^\bullet \in (\mathbb{Z}^n)^{Q_0}$, we have $Q_{\mu^\bullet} = Q_{\mu_1^\bullet} \cdots Q_{\mu_n^\bullet} \cdot 1$ where the operators Q_{d^\bullet} are given by

the generating series

$$\begin{aligned} Q(u^\bullet) &= \sum_{d^\bullet \in \mathbb{Z}^{Q_0}} u^{d^\bullet} Q_{d^\bullet} = \Psi H(u^\bullet) \Psi^{-1} \\ &= \Omega \left[\sum_{(i,j) \in Q_0^2} \frac{u^{(j)}}{u^{(i)}} \hat{A}_{ij} \right] \\ &\quad \times \Omega \left[\sum_{(i,j) \in Q_0^2} u^{(i)} (\delta_{ij} - A'_{ij}) X^{(j)} \right] \Omega \left[- \sum_{j \in Q_0} (u^{(j)})^{-1} X^{(j)} \right]^\perp \end{aligned}$$

where A' is the matrix (8) corresponding to the opposite quiver Q' .

Remark 13. Shoji [Sh3, Theorem 6.4] showed that for the cyclic quiver in Example 4, the matrix $(L_{\lambda^\bullet, \mu^\bullet})$ expressing Q_{λ^\bullet} in terms of s_{μ^\bullet} is lower triangular.

On the other hand, if we take the cyclic quiver on 3 vertices but take the acyclic subquiver to consist only of the edge $(2, 0)$, then

$$\begin{aligned} Q_{((1), \emptyset, \emptyset)} &= s_{((1), \emptyset, \emptyset)} - q_{12} q_{20} s_{(\emptyset, (1), \emptyset)} \\ Q_{(\emptyset, (1), \emptyset)} &= -q_{01} s_{((1), \emptyset, \emptyset)} + s_{(\emptyset, (1), \emptyset)} \\ Q_{(\emptyset, \emptyset, (1))} &= -q_{12} s_{(\emptyset, (1), \emptyset)} + s_{(\emptyset, \emptyset, (1))} \end{aligned}$$

shows that no ordering makes these Q functions expand triangularly with the s functions.

It appears that if Q itself is acyclic then the expansion is triangular.

4. QUIVER HALL-LITTLEWOOD P -FUNCTIONS

Let $\{P_{\mu^\bullet} \mid \mu^\bullet \in \mathbb{Y}^{Q_0}\}$ be the basis of Λ_Q dual to $\{H_{\mu^\bullet}\}$ under the Hall scalar product. Thus for any $\lambda^\bullet \in \mathbb{Y}^{Q_0}$ we have the expansion

$$(10) \quad s_{\lambda^\bullet}[X^\bullet] = \sum_{\mu^\bullet \in \mathbb{Y}^{Q_0}} \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) P_{\mu^\bullet}[X^\bullet; q_{Q_1}].$$

We note also that, by Section 3.2, the bases $\{Q_{\mu^\bullet}\}$ and $\{P_{\mu^\bullet}\}$ are dual under $\langle \cdot, \cdot \rangle_Q$.

The purpose of this section is to give a different description of the images of P_{μ^\bullet} under the projection $X^\bullet \mapsto x_1^\bullet + \cdots + x_n^\bullet$ to a finite set of variables. Our approach is motivated by the classical construction of Hall-Littlewood functions [Mac] and Shoji's definition of Kostka-Shoji polynomials [Sh1].

4.1. A family of Laurent polynomials. Let n be a positive integer and define $\text{Pol}_n = \text{Pol}_{Q,n} = S[(x_k^{(i)})^{\pm 1} \mid i \in Q_0, 1 \leq k \leq n]$ where $S = \mathbb{Q}[q_{ij} \mid (i, j) \in Q_1]$. Define for any quiver Q and any (not necessarily acyclic) subquiver \hat{Q} , the Laurent polynomials

$$\begin{aligned} R_{\mu^\bullet}^{Q, \hat{Q}}(x; q_{Q_1}) &= D_{w_0} \left(x^{\mu^\bullet} \Omega \left[- \sum_{k > \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] \Omega \left[- \sum_k (x_k^\bullet) \hat{A}(x_k^\bullet)^* \right] \right) \\ &= \sum_{w^\bullet} w^\bullet \left(x^{\mu^\bullet} \Omega \left[\sum_{k > \ell} x_k^\bullet (I - A)(x_\ell^\bullet)^* \right] \Omega \left[- \sum_k (x_k^\bullet) \hat{A}(x_k^\bullet)^* \right] \right) \end{aligned}$$

for all $\mu^\bullet \in X_+^{Q_0}$. Here a matrix expression $x_k^\bullet A(x_\ell^\bullet)^*$ is shorthand for the sum $\sum_{i,j \in Q_0} x_k^{(i)} A_{ij}(x_\ell^{(j)})^{-1}$. We write $R_{\mu^\bullet}(x; q_{Q_1})$ when Q, \hat{Q} are clear from context.

We observe that $R_{\mu^\bullet}(x; 0) = s_{\mu^\bullet}(x)$. Hence the $R_{\mu^\bullet}(x; q_{Q_1})$ for $\mu^\bullet \in X_+^{Q_0}$ form a basis of $\text{Sym}_n \otimes_S F$, where $\text{Sym}_n = \text{Pol}_n^{S_n^{Q_0}}$.

Remark 14. In the case of a cyclic quiver, the Laurent polynomials defined above are similar but not identical to Shoji's polynomials $R_{\mu^\bullet}^\pm(z; \mathbf{t})$ [Sh3, (4.1.2)]. Let Q be the cyclic quiver of Example 4. Let Q' be the opposite quiver.

Shoji's $R_{\mu^\bullet}^\pm(z; \mathbf{t})$ are polynomials in the variables $\{z_k^{(i)} : i \in Q_0, 1 \leq k \leq n\}$ with coefficients in $\mathbb{Q}[\mathbf{t}]$ where $\mathbf{t} = \{t_1, \dots, t_r\}$. We make the following identification with our variables and parameters: $z_k^{(i)} = x_k^{(i-1)}$ and $t_i = q_{i-1, i}$. For any $\mu^\bullet \in X_+^{Q_0}$, let $\mu^{\bullet-1}$ be the shifted Q_0 -tuple of dominant weights whose entry at $i \in Q_0$ is $\mu^{(i-1)}$.

Suppose that \hat{Q}' is the acyclic subquiver of Q' obtained by deleting the edge $(0, r-1)$, and let \hat{A}' be its arrow matrix. Up to a scalar factor depending on μ^\bullet which is a polynomial in q_{Q_1} with constant term 1, Shoji's polynomials $R_{\mu^{\bullet-1}}^-(z; \mathbf{t})$ are given by

$$D_{w_0} \left(x^{\mu^\bullet} \Omega \left[- \sum_{k > \ell} x_k^\bullet A'(x_\ell^\bullet)^* \right] \Omega \left[- \sum_k^* x_k^\bullet \hat{A}'(x_k^\bullet)^* \right] \right)$$

where $\sum_k^* x_k^\bullet \hat{A}'(x_k^\bullet)^*$ is the sum of terms $q_{i, i+1} x_k^{(i+1)} / x_k^{(i)}$ such that $\mu_k^{(i)} \neq 0$. Our

Laurent polynomials $R_{\mu^\bullet}^{Q', \hat{Q}'}(x; q_{Q_1})$ are given by this formula but without this restriction on the sum. For example, when $r = 3$, $n = 1$, $\mu^{(0)} = (0)$, $\mu^{(1)} = (1)$, and $\mu^{(2)} = (0)$ we have

$$\begin{aligned} R_{\mu^{\bullet-1}}^-(z; \mathbf{t}) &= z_1^{(2)}(1 - t_2 z_1^{(3)} / z_1^{(2)}) = x_1^{(1)}(1 - q_{12} x_1^{(2)} / x_1^{(1)}) \\ R_{\mu^\bullet}^{Q', \hat{Q}'}(x; q_{Q_1}) &= x_1^{(1)}(1 - q_{01} x_1^{(1)} / x_1^{(0)})(1 - q_{12} x_1^{(2)} / x_1^{(1)}). \end{aligned}$$

For the subquiver \tilde{Q} of Q containing *only* the edge $(r-1, 0)$, Shoji's polynomials $R_{\mu^{\bullet-1}}^+(z; \mathbf{t})$ are up to a scalar factor given by

$$D_{w_0} \left(x^{\mu^\bullet} \Omega \left[- \sum_{k > \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] \Omega \left[- \sum_k^* x_k^\bullet \tilde{A}(x_k^\bullet)^* \right] \right),$$

with the restricted sum running over terms $q_{0, r-1} x_k^{(r-1)} / x_k^{(0)}$ with $\mu_k^{(0)} \neq 0$. The $R_{\mu^\bullet}^{Q, \tilde{Q}}(x; q_{Q_1})$ are given by this formula without the restriction on the sum.

4.2. Auxiliary Hall pairing. Define a pairing $(\cdot, \cdot) : \text{Pol}_n \times \text{Pol}_n \rightarrow S$ by

$$\begin{aligned} (f, g) &= |S_n^{Q_0}|^{-1} \text{CT}(f \Delta^\bullet (g \Delta^\bullet)^*) \\ &= |S_n^{Q_0}|^{-1} \text{CT}(f g^* \prod_{\substack{i \in Q_0 \\ k \neq \ell}} (1 - x_\ell^{(i)} / x_k^{(i)})) \end{aligned}$$

where CT is the constant term in $\{x_k^{(i)} : i \in Q_0, 1 \leq k \leq n\}$. Restricted to Sym_n , this is simply the Hall scalar product:

Lemma 15. *For all $\lambda^\bullet, \mu^\bullet \in X_+^{Q_0}$ we have $(s_{\lambda^\bullet}(x), s_{\mu^\bullet}(x)) = \delta_{\lambda^\bullet, \mu^\bullet}$.*

Proof. We have $(s_{\lambda^\bullet}(x), s_{\mu^\bullet}(x)) = |S_n^{Q_0}|^{-1} \text{CT}(J^\bullet(x^{\lambda^\bullet + \rho^\bullet}) J^\bullet(x^{\mu^\bullet + \rho^\bullet})^*)$. \square

4.3. Auxiliary quiver pairing. Let \bar{Q} be the quiver obtained from Q by deleting all loop edges (if any), and let \bar{A} be its arrow matrix. Thus \bar{A} has 0's along the diagonal and agrees with A everywhere else. We define a pairing $(\cdot, \cdot)_Q : \text{Pol}_n \times \text{Pol}_n \rightarrow \mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$ by

$$(f, g)_Q = |S_n^{Q_0}|^{-1} \text{CT} \left(f(x)g(x)^* \Omega \left[\sum_{k \neq \ell} (x_k^\bullet)^* (A - I)x_\ell^\bullet \right] \Omega \left[\sum_k (x_k^\bullet)^* \bar{A}x_k^\bullet \right] \right).$$

The factors inside CT in this definition are chosen to ensure that the pairing is $S_n^{Q_0}$ -invariant.

Observe that $(\cdot, \cdot)_Q$ coincides with (\cdot, \cdot) when all the parameters q_{Q_1} are set to 0; this implies that $(\cdot, \cdot)_Q$ is nondegenerate with Gram matrix invertible over $\mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$.

Remark 16. In the case of the Jordan quiver with $Q_0 = \{0\}$, $(\cdot, \cdot)_Q$ is the finite-variable Macdonald pairing [Mac, VI (9.3)] under a suitable specialization (the Macdonald parameter q is set to 0, while our quiver parameter q_{00} is the Macdonald t parameter).

We observe that (f, g) and $(f, g)_Q$ are well-defined as elements of $\mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$ for any $f, g \in \text{Pol}_n \otimes_S \mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$. These pairings enable one to related the Laurent polynomials of Section 4.1 to the quiver Hall-Littlewood series $\chi_{\mu^\bullet}^{Q, \hat{Q}}$ as follows.

Proposition 17. *For any $\lambda^\bullet, \mu^\bullet \in X_+^{Q_0}$*

$$(11) \quad \mathcal{K}_{\lambda^\bullet \mu^\bullet}(q_{Q_1}) = (s_{\lambda^\bullet}, \chi_{\mu^\bullet}^{Q, \hat{Q}}) = (s_{\lambda^\bullet}, R_{\mu^\bullet}^{Q, \bar{Q} - \hat{Q}})_Q.$$

Proof. The first equality holds by (4) and Lemma 15. Now $|S_n^{Q_0}|(s_{\lambda^\bullet}, \chi_{\mu^\bullet})$ is equal to

$$\text{CT} \left(s_{\lambda^\bullet}(x) \left(D_{w_0^\bullet} x^{\mu^\bullet} \Omega \left[\sum_k x_k^\bullet \hat{A}(x_k^\bullet)^* \right] \Omega \left[\sum_{k < \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] \right)^* \Omega \left[- \sum_{k \neq \ell} (x_k^\bullet)^* I x_\ell^\bullet \right] \right).$$

We need only write

$$\begin{aligned} \Omega \left[\sum_k x_k^\bullet \hat{A}(x_k^\bullet)^* \right] \Omega \left[\sum_{k < \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] &= \Omega \left[\sum_k x_k^\bullet \bar{A}(x_k^\bullet)^* \right] \Omega \left[\sum_{k \neq \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] \\ &\quad \times \Omega \left[- \sum_k x_k^\bullet (\bar{A} - \hat{A})(x_k^\bullet)^* \right] \Omega \left[- \sum_{k > \ell} x_k^\bullet A(x_\ell^\bullet)^* \right] \end{aligned}$$

and observe that the product of the first two factors on the right-hand side is $S_n^{Q_0}$ -invariant. Hence we can commute it past $D_{w_0^\bullet}$ to obtain the second equality. \square

4.4. Dual expansion. For any $\lambda^\bullet \in X_+^{Q_0}$, let us define $P_{\lambda^\bullet}(x; q_{Q_1})$ by duality:

$$(P_{\lambda^\bullet}, R_{\mu^\bullet}^{Q, \bar{Q} - \hat{Q}})_Q = \delta_{\lambda^\bullet \mu^\bullet}.$$

Then the P_{λ^\bullet} form a $\mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$ -module basis for $\text{Sym}_n \otimes_S \mathbb{Q}[[q_{ij} \mid (i, j) \in Q_1]]$ and we obtain

$$s_{\lambda^\bullet}(x) = \sum_{\mu^\bullet \in X_+^{Q_0}} \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) P_{\mu^\bullet}(x; q_{Q_1}).$$

for all $\lambda^\bullet \in X_+^{Q_0}$. We have the following improvement when $\lambda^\bullet \in \mathbb{Y}_n^{Q_0}$:

Proposition 18. *For all $\lambda^\bullet \in \mathbb{Y}_n$*

$$(12) \quad s_{\lambda^\bullet}(x) = \sum_{\mu^\bullet \in \mathbb{Y}_n} \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) P_{\mu^\bullet}(x; q_{Q_1}),$$

and for any $\mu^\bullet \in \mathbb{Y}_n$, $P_{\mu^\bullet}(x; q_{Q_1})$ is polynomial in $\{x_k^{(i)} \mid i \in Q_0, 1 \leq k \leq n\}$.

Proof. Recall that $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ vanishes unless $\lambda^\bullet \geq \mu^\bullet$. The following observation then proves (12): if $\lambda^\bullet \in \mathbb{Y}_n$, $\mu^\bullet \in X_+^{Q_0}$, and $\lambda^\bullet \geq \mu^\bullet$, then it is necessarily the case that $\mu^\bullet \in \mathbb{Y}_n$. Inverting the matrix of quiver Kostka-Shoji polynomials $(\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}))_{\lambda^\bullet, \mu^\bullet \in \mathbb{Y}_n^{Q_0}}$, which is unitriangular in the dominance order, we obtain that $P_{\mu^\bullet}(x; q_{Q_1})$ is a polynomial for $\mu^\bullet \in \mathbb{Y}_n$, i.e., no negative powers of the $x_k^{(i)}$ appear. \square

Comparing with the symmetric function expansion (10), we deduce:

Corollary 19. *For all $\mu^\bullet \in \mathbb{Y}_n^{Q_0}$, $P_{\mu^\bullet}(x; q_{Q_1})$ is equal to the image of the symmetric function $P_{\mu^\bullet}[X; q_{Q_1}]$ under the projection to finitely-many variables given by $X^\bullet \mapsto x_1^\bullet + \cdots + x_n^\bullet$.*

5. MORRIS RECURRENCE

There is a recurrence for $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ based on the Q_0 -fold product of induction from B to P to GL_n where P has Levi subgroup $GL_1 \times GL_{n-1}$. Let $w_0^{(n-1)\bullet}$ be the long element in the subgroup $(S_1 \times S_{n-1})^{Q_0} \subset S_n^{Q_0}$ generated by the reflections $s_2^{(i)}$ through $s_{n-1}^{(i)}$ where $s_j^{(i)}$ is the simple reflection in the i -th copy of S_n in $S_n^{Q_0}$.

Recall that a Q_0 -tuple $\mu^\bullet \in \mathbb{Y}_n^{Q_0}$ of partitions with at most n parts, can be viewed as a sequence of n dimension vectors. We write μ_1^\bullet for the first dimension vector and $\hat{\mu}^\bullet$ for the sequence of the remaining $n-1$ dimension vectors.

Let $R_{+,n-1}^{Q,\hat{Q}}$ be the roots in $R_+^{Q,\hat{Q}}$ of the form $e_k \otimes \epsilon^{(i)} - e_\ell \otimes \epsilon^{(j)}$ for $2 \leq k < \ell \leq n$ and $(i, j) \in Q_1$ and those of the form $e_k \otimes (\epsilon^{(i)} - \epsilon^{(j)})$ for $2 \leq k \leq n$ and $(i, j) \in \hat{Q}_1$. The roots in $R_+^{Q,\hat{Q}} \setminus R_{+,n-1}^{Q,\hat{Q}}$ are those of the form $e_1 \otimes \epsilon^{(i)} - e_\ell \otimes \epsilon^{(j)}$ for $(i, j) \in Q_1$ and $2 \leq \ell \leq n$, and $e_1 \otimes (\epsilon^{(i)} - \epsilon^{(j)})$ for $(i, j) \in \hat{Q}_1$.

Let $B_{\mu^\bullet} = x^{\mu^\bullet} B(x; q_{Q_1})$ and $B_{\hat{\mu}^\bullet} = \hat{x}^{\hat{\mu}^\bullet} B(\hat{x}; q_{Q_1})$ where $\hat{x} = (x_2^\bullet, \dots, x_n^\bullet)$. Since $D_{w_0}^{(n-1)\bullet}$ commutes with $(S_1 \times S_{n-1})^{Q_0}$ -invariant elements, we have

$$\begin{aligned} \chi_{\mu^\bullet} &= D_{w_0^\bullet} B_{\mu^\bullet} = D_{w_0^\bullet} D_{w_0^{(n-1)\bullet}} B_{\mu^\bullet} \\ &= D_{w_0^\bullet} D_{w_0^{(n-1)\bullet}} x_1^{\mu_1^\bullet} \prod_{(i,j) \in \hat{Q}_1} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_1^{(j)}}\right)^{-1} \prod_{\substack{(i,j) \in Q_1 \\ 2 \leq \ell \leq n}} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_\ell^{(j)}}\right)^{-1} B_{\hat{\mu}^\bullet} \\ &= D_{w_0^\bullet} x_1^{\mu_1^\bullet} \prod_{(i,j) \in \hat{Q}_1} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_1^{(j)}}\right)^{-1} \prod_{\substack{(i,j) \in Q_1 \\ 2 \leq \ell \leq n}} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_\ell^{(j)}}\right)^{-1} D_{w_0^{(n-1)\bullet}} B_{\hat{\mu}^\bullet} \\ &= D_{w_0^\bullet} x_1^{\mu_1^\bullet} \prod_{(i,j) \in \hat{Q}_1} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_1^{(j)}}\right)^{-1} \prod_{\substack{(i,j) \in Q_1 \\ 2 \leq \ell \leq n}} \left(1 - q_{ij} \frac{x_1^{(i)}}{x_\ell^{(j)}}\right)^{-1} \chi_{\hat{\mu}^\bullet}. \end{aligned}$$

For functions $b : Q_1 \rightarrow \mathbb{Z}_{\geq 0}$ we define the monomial q^b and virtual dimension vectors $\text{wt}^+(b), \text{wt}^-(b), \text{wt}(b) \in \mathbb{Z}^{Q_0}$ by

$$\begin{aligned} q^b &= \prod_{(i,j) \in Q_1} q_{ij}^{b(i,j)} \\ \text{wt}^+(b) &= \sum_{(i,j) \in Q_1} b(i,j) \epsilon^{(i)} \\ \text{wt}^-(b) &= \sum_{(i,j) \in Q_1} b(i,j) \epsilon^{(j)} \\ \text{wt}(b) &= \text{wt}^+(b) - \text{wt}^-(b). \end{aligned}$$

We have the $(S_1 \times S_{n-1})^{Q_0}$ -invariant expression

$$\begin{aligned} D_{w_0^{(n-1)\bullet}} B_{\mu^\bullet} &= x_1^{\mu_1^\bullet} \sum_{b: \hat{Q}_1 \rightarrow \mathbb{Z}_{\geq 0}} q^b x_1^{\text{wt}(b)} \sum_{c: Q_1 \rightarrow \mathbb{Z}_{\geq 0}} q^c x_1^{\text{wt}^+(c)} \prod_{(i,j) \in Q_1} s_{c(i,j)} [\hat{x}^{(j)*}] \\ &\quad \times \sum_{\gamma^\bullet \in \mathbb{Y}_{n-1}^{Q_0}} \mathcal{K}_{\gamma^\bullet, \mu^\bullet}(q_{Q_1}) s_{\gamma^\bullet}(\hat{x}). \end{aligned}$$

Lemma 20. For $\lambda \in \mathbb{Y}_n$, $f(z_1) \in \mathbb{Q}[z_1^\pm]$ and $g \in \mathbb{Q}[z_2^\pm, \dots, z_n^\pm]$ that is invariant under $S_1 \times S_{n-1}$ and writing $\hat{z} = (z_2, \dots, z_n)$, we have for D_{w_0} acting on $z = (z_1, \dots, z_n)$:

$$[s_\lambda(z)] D_{w_0} f(z_1) g(\hat{z}) = \sum_{w \in S_n / (S_1 \times S_{n-1})} (-1)^w [z_1^{m(w)}] f(z_1) [s_{\beta(w)}(\hat{z})] g(\hat{z})$$

where $w^{-1}(\lambda + \rho) - \rho = (m(w), \beta(w))$ with $m(w) \in \mathbb{Z}$ and $\beta(w) \in \mathbb{Y}_{n-1}$.

For $\lambda^\bullet \in \mathbb{Y}_n^{Q_0}$, we apply Lemma 20 in each copy of \mathbb{Z}^n , with w^\bullet running over Q_0 tuples of minimum coset representatives in $S_n / (S_1 \times S_{n-1})$. Let $w^{\bullet-1}(\lambda^\bullet + \rho^\bullet) - \rho^\bullet = (m^\bullet(w^\bullet), \beta^\bullet(w^\bullet))$ with $m^\bullet(w^\bullet) \in \mathbb{Z}^{Q_0}$ and $\beta^\bullet(w^\bullet) \in \mathbb{Y}_{n-1}^{Q_0}$. Let $\langle \cdot, \cdot \rangle_{GL_n^{Q_0}}$ be the inner product on $GL_n^{Q_0}$ -characters with irreducible characters orthonormal. Letting b and c run over functions $b : \hat{Q}_1 \rightarrow \mathbb{Z}_{\geq 0}$, $c : Q_1 \rightarrow \mathbb{Z}_{\geq 0}$, and $\gamma^\bullet \in \mathbb{Y}_{n-1}^{Q_0}$ we obtain

the following Morris-type recurrence for the quiver Kostka-Shoji polynomials:

$$\begin{aligned}
\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) &= [s_{\lambda^\bullet}(x)] D_{w_0^\bullet} D_{w_0^{(n-1)}^\bullet} B_{\mu^\bullet} \\
&= \sum_{w^\bullet} (-1)^{w^\bullet} \sum_{b, c, \gamma^\bullet} q^{b+c} [x_1^{m^\bullet(w^\bullet)}] x_1^{\mu_1^\bullet + \text{wt}(b) + \text{wt}^+(c)} \\
&\quad \times [s_{\beta^\bullet(w^\bullet)}(\hat{x})] \prod_{(i,j) \in Q_1} s_{c(i,j)}(\hat{x}^{(j)*}) s_{\gamma^\bullet}(\hat{x}) K_{\gamma^\bullet, \hat{\mu}^\bullet}(q_{Q_1}) \\
&= \sum_{w^\bullet} (-1)^{w^\bullet} \sum_{b, c, \gamma^\bullet} q^{b+c} \langle s_{\beta^\bullet(w^\bullet)}(\hat{x}) \prod_{(i,j) \in Q_1} s_{c(i,j)}(\hat{x}^{(j)}) , s_{\gamma^\bullet}(\hat{x}) \rangle_{GL_{n-1}^{Q_0}} \\
&\quad \times K_{\gamma^\bullet, \hat{\mu}^\bullet}(q_{Q_1})
\end{aligned}$$

where in the last equality the inner sum is restricted to b, c such that

$$m^\bullet(w^\bullet) = \mu_1^\bullet + \text{wt}(b) + \text{wt}^+(c).$$

6. COMBINATORICS FOR CYCLIC QUIVER

For this section let Q and \hat{Q} be the r -vertex cyclic quiver and acyclic subquiver of Example 4. Let n be a fixed positive integer. Vertex indices are taken modulo r .

6.1. Reduced Kostka-Shoji polynomials. An r -multipartition is a Q_0 -tuple $\mu^\bullet = (\mu^{(0)}, \dots, \mu^{(r-1)}) \in \mathbb{Y}_n^{Q_0}$ of partitions with at most n parts. This multipartition has size $|\mu^\bullet| = \sum_{i \in Q_0} |\mu^{(i)}|$. $\mathbb{Y}_n^{Q_0}$ has the dominance partial order; see §2.7.

Lemma 21. *There is a unique Laurent monomial of the form $\prod_{i=0}^{r-2} q_{i,i+1}^{a_i}$ with $a_i \in \mathbb{Z}$ and a unique polynomial $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}^{\text{red}}(q) \in \mathbb{Z}[q]$ in*

$$q = q_{01} q_{12} \cdots q_{r-2, r-1} q_{r-1, 0}$$

such that

$$(13) \quad \mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1}) = \left(\prod_{i=0}^{r-2} q_{i,i+1}^{a_i} \right) \mathcal{K}_{\lambda^\bullet, \mu^\bullet}^{\text{red}}(q).$$

We call $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}^{\text{red}}(q)$ the reduced Kostka-Shoji polynomial.

Proof. By (5), $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}(q_{Q_1})$ is a finite signed sum of monomials $\prod_{\alpha} q_{\alpha}^{m(\alpha)}$ with the product over $\alpha \in R_+^{Q, \hat{Q}}$, where $m : R_+^{Q, \hat{Q}} \rightarrow \mathbb{Z}_{\geq 0}$ satisfies (6) for some $w^\bullet \in S_n^{Q_0}$. By (3) the monomial in the arrow variables is invariant under the action of $S_n^{Q_0}$. Therefore every such monomial $\prod_{i=0}^{r-1} q_{i,i+1}^{a_i}$ must satisfy the property that $\text{pr}(\lambda^\bullet - \mu^\bullet) = \sum_{i=0}^{r-1} a_i (\epsilon^{(i)} - \epsilon^{(i+1)})$. Thus all exponent vectors (a_0, \dots, a_{r-1}) must be congruent modulo the vector $(1, 1, \dots, 1) \in \mathbb{Z}^{Q_0}$. We factor out the unique such monomial whose exponent vector has $a_{r-1} = 0$. The Lemma follows. \square

Example 22. The monomial of Lemma 21 is given explicitly by taking partial sums. For example consider $r = 4$ and suppose we want the arrow variable monomial for the weight $\text{pr}(\lambda^\bullet - \mu^\bullet) = (-3, 1, -2, 4)$. Then $a_0 = -3$, $a_1 = -3 + 1 = -2$ and $a_2 = -3 + 1 - 2 = -4$ and the monomial is $q_{01}^{-3} q_{12}^{-2} q_{23}^{-4}$.

Example 23. Let $r = 2$ and $\lambda^\bullet = ((1), (2))$, $\mu^\bullet = ((1, 1), (1))$. We have $j(\lambda^\bullet) - j(\mu^\bullet) = (1, 2, 0, 0) - (1, 1, 1, 0) = (0, 1, -1, 0)$. So $\text{pr}(j(\lambda^\bullet) - j(\mu^\bullet)) = (-1, 1)$ has arrow monomial q_{01}^{-1} . With $q = q_{01}q_{10}$ we have

$$\mathcal{K}_{((1),(2)),((1,1),(1))}(q_{01}, q_{10}) = q_{10} = q_{01}^{-1}(q_{01}q_{10})$$

$$\mathcal{K}_{((1),(2)),((1,1),(1))}^{\text{red}}(q) = q.$$

6.2. Multitableaux and cascading catabolism type. Below we will give a conjecture for $\mathcal{K}_{\lambda^\bullet, \mu^\bullet}^{\text{red}}(q)$ as a weighted sum over a certain set of multitableaux. A multitableau of shape $\lambda^\bullet \in \mathbb{Y}_n^{Q_0}$ is a sequence $T^\bullet = (T^{(0)}, \dots, T^{(r-1)})$ of semistandard tableaux with $T^{(i)}$ of shape $\lambda^{(i)}$ using the alphabet $1, 2, \dots, n$. For $1 \leq k \leq n$ let $m_k(T^\bullet)$ be the dimension vector whose i -th component is the number of letters k in $T^{(i)}$.

Example 24. With $r = 3$ and $n = 4$,

$$T^\bullet = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & & & \\ \hline 3 & 3 & & & \\ \hline 4 & 4 & & & \\ \hline \end{array}$$

is a multitableau of shape $((4, 4, 0, 0), (5, 5, 0, 0), (5, 2, 2, 2))$ with $m_1(T^\bullet) = (3, 3, 2)$.

The *cascading catabolism* $\text{ccat}(T^\bullet)$ is the multitableau resulting from the following algorithm applied to a multitableau T^\bullet .

- For i from 0 to $r - 1$, remove the first row from the i -th tableau. Remove the 1s from the removed word and Schensted column insert the remainder into the $i + 1$ -th tableau. For $i = r - 1$, insert into the 0-th tableau.

Example 25. For the above T^\bullet , after the $i = 0$ step we have

$$\begin{array}{|c|c|c|c|} \hline 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 & & & \\ \hline 3 & 3 & & & \\ \hline 4 & 4 & & & \\ \hline \end{array}$$

After the $i = 1$ step we have

$$\begin{array}{|c|c|c|c|} \hline 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 3 & & & \\ \hline 3 & 3 & & & & \\ \hline 4 & 4 & & & & \\ \hline \end{array}$$

After the $i = 2$ step we have

$$\text{ccat}(T^\bullet) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 2 & 2 & 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 2 & 2 & 3 \\ \hline 3 & 3 & \\ \hline 4 & 4 & \\ \hline \end{array}$$

The *cascading catabolism type* $\text{cctype}(T^\bullet)$ is the sequence of dimension vectors

$$(14) \quad \text{cctype}(T^\bullet) = (m_1(T^\bullet), m_2(\text{ccat}(T^\bullet)), m_3(\text{ccat}^2(T^\bullet)), \dots, m_n(\text{ccat}^{n-1}(T^\bullet))).$$

During the computation of the i -th cascading catabolism we remove the letters i (which are minimal in the existing letters) rather than 1s.

Example 26. We have

$$\text{ccat}^2(T^\bullet) = \begin{array}{|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline 3 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 4 & 4 \\ \hline \end{array}$$

$$\text{ccat}^3(T^\bullet) = \begin{array}{|c|c|c|} \hline 4 & 4 & 4 \\ \hline \end{array} \otimes \emptyset \otimes \begin{array}{|c|c|} \hline 4 & 4 \\ \hline \end{array}$$

$$\text{cctype}(T^\bullet) = ((3, 3, 2), (5, 1, 2), (5, 1, 2), (3, 0, 2)).$$

Say that a multitableau is μ^\bullet -cascade-catabolizable if that there is componentwise dominance $\text{cctype}(T^\bullet) \supseteq \mu^\bullet$, that is,

$$(15) \quad m_k(\text{ccat}^{k-1}(T^\bullet)) \supseteq \mu_k^\bullet \quad \text{for all } 1 \leq k \leq n$$

where \supseteq is the usual dominance order on the weight lattice \mathbb{Z}^{Q_0} of $GL(\mathbb{C}^{Q_0})$.

Example 27. For

$$\mu^\bullet = ((2, 2, 2, 2), (3, 3, 3, 0), (3, 3, 3, 3)) \in \mathbb{Y}_4^{Q_0}$$

we see that T^\bullet is μ^\bullet -cascade catabolizable since μ^\bullet is the sequence of dimension vectors

$$((2, 3, 3), (2, 3, 3), (2, 3, 3), (2, 0, 3))$$

and $(3, 3, 2) \supseteq (2, 3, 3)$, $(5, 1, 2) \supseteq (2, 3, 3)$, $(5, 1, 2) \supseteq (2, 3, 3)$, and $(3, 0, 2) \supseteq (2, 0, 3)$.

The following multitableau is not μ^\bullet -cascade catabolizable.

$$S^\bullet = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & & & \\ \hline 3 & 3 & & & \\ \hline 4 & 4 & & & \\ \hline \end{array}$$

We have

$$\text{ccat}(S^\bullet) = \begin{array}{|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 4 & 4 & 4 \\ \hline 3 & & & & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 3 & 3 \\ \hline 3 & 3 & & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 3 & 3 & \\ \hline 4 & 4 & \\ \hline \end{array}$$

$$\text{ccat}^2(S^\bullet) = \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 4 & 4 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 3 & 3 & 4 \\ \hline 4 & 4 & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 4 & 4 & \\ \hline \end{array}$$

$$\text{ccat}^3(S^\bullet) = \emptyset \otimes \begin{array}{|c|c|} \hline 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 4 & 4 & 4 \\ \hline \end{array}$$

so that

$$\text{cctype}(S^\bullet) = ((2, 2, 3), (3, 2, 3), (3, 2, 3), (0, 2, 3))$$

The failure occurs for the 4-th dimension vector: $(0, 2, 3) \not\supseteq (2, 0, 3)$.

Conjecture 28. For the cyclic quiver

$$(16) \quad \mathcal{K}_{\lambda^\bullet \mu^\bullet}^{\text{red}}(q) = \sum_{T^\bullet} q^{\text{charge}(T^{(0)} T^{(1)} \dots T^{(r-1)})}$$

where the sum runs over the set of μ^\bullet -cascade-catabolizable multitableaux T^\bullet of shape λ^\bullet and the power of q is the charge of Lascoux and Schützenberger [LS] applied to the concatenation of the words of the tableaux in T^\bullet .

Example 29. For $\lambda^\bullet = ((1), (2))$ and $\mu^\bullet = ((1, 1), (1))$ as in Example 23, there are two potential multitableaux of shape λ^\bullet and the correct total number of 1's and 2's, namely:

$$\boxed{1} \otimes \boxed{1 \ 2} \quad \text{and} \quad \boxed{2} \otimes \boxed{1 \ 1}$$

Only the first multitableau is μ^\bullet -cascade catabolizable. This has charge 1 and we recover $\mathcal{K}_{\lambda^\bullet \mu^\bullet}^{\text{red}}(q) = q$. This agrees with [Sh3, Table 2].

For $r = 1$, μ^\bullet is a single partition μ and a tableau is μ^\bullet -cascade catabolizable if and only if it has weight μ . Thus the conjecture reduces to the theorem of [LS] that gives the Kostka-Foulkes polynomial $K_{\lambda\mu}(q)$ as the sum over semistandard tableaux of shape λ and weight μ with the charge statistic.

Theorem 30. *Conjecture 28 holds for $n = 2$ and any r .*

We have an algorithm which appears to produce a cancelling involution when the combinatorial formula of Conjecture 28 is plugged into the Morris recurrence. However we are not able to verify in general that our putative cancelling map has image inside the set to be cancelled, due to the trickiness of cascade-catabolizability. For $n = 2$ the cascade-catabolizability condition is very simple as it is only needed for multitableaux with a single kind of letter; in this case we can prove our algorithm produces a well-defined sign-reversing involution, thereby establishing Conjecture 28 for $n = 2$.

7. PARABOLIC ANALOGUES

In a separate work [OS] we shall consider the parabolic analogues of quiver symmetric functions and Kostka-Shoji polynomials. This amounts to replacing the Lie algebra \mathfrak{n} of the unipotent radical of the Borel subgroup, by the corresponding construction for a parabolic subgroup of GL_n . For a single loop we obtain the parabolic analogue of H Hall-Littlewood and Kostka-Foulkes polynomials [SZ1] coming from the Springer resolution of a general nilpotent adjoint orbit closure instead of the full nullcone. In the single-loop context these parabolic Kostka polynomials and their creation operators have been useful for studying affine type A Kirillov-Reshetikhin characters and k -Schur functions [LLM] [LM]. In [LM] the k -split symmetric functions, which are a subfamily of the parabolic H Hall-Littlewood symmetric functions, are used to give a definition of the k -Schur function with grading parameter.

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